1. **In this question you should show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

(a) Using algebra, find all solutions of the equation

$$3x^3 - 17x^2 - 6x = 0$$

(3)

(b) Hence find all real solutions of

$$3(y-2)^6 - 17(y-2)^4 - 6(y-2)^2 = 0$$

(3)

a)

$3x^3 - 17x^2 - 6x = 0$

$\Rightarrow x(3x^2 - 17x - 6) = 0$ (1)

$\Rightarrow x(3x+1)(x-6) = 0$ (1)

So, the solutions are $x = 0, -\frac{1}{3}, 6$ (1)

b)

Let $n = (y-2)^2$, then

$3(y-2)^6 - 17(y-2)^4 - 6(y-2)^2 = 0$ (1)

$3n^3 - 17n^2 - 6n = 0$ has the solutions :

$\downarrow$

$n = 0, -\frac{1}{3}, 6$ from part (a)

except $n \neq -\frac{1}{3}$ as $n \geqslant 0$ (as it is squared)

$\Rightarrow (y-2)^2 = 0$ and $(y-2)^2 = 6$

which gives the solutions

$y = 2$ (1) and $y = 2 \pm \sqrt{6}$ (1)

2.

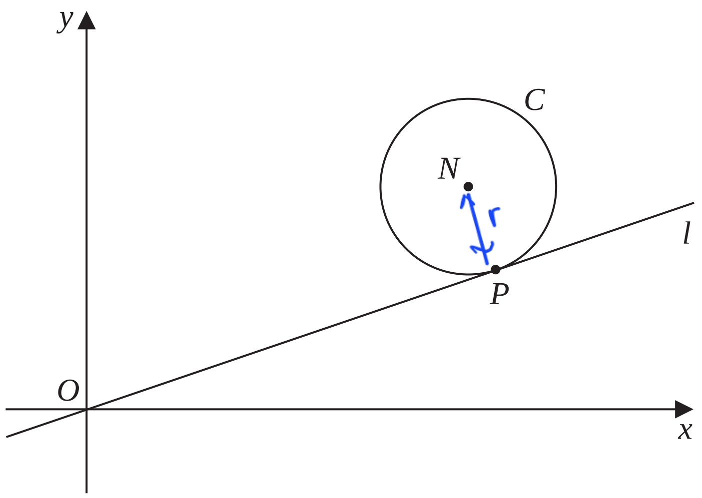

Figure 4

Figure 4 shows a sketch of a circle C with centre $N(7, 4)$

The line l with equation $y = \frac{1}{3}x$ is a tangent to C at the point P.

Find

$m_l = \frac{1}{3}$

(a) the equation of line PN in the form $y = mx + c$, where m and c are constants,

(2)

(b) an equation for C.

(4)

The line with equation $y = \frac{1}{3}x + k$, where k is a non-zero constant, is also a tangent to C.

(c) Find the value of k.

$m_{PN} = \frac{-1}{m_l}$

(3)

a) PN is perpendicular to l, so $m = -3$, and as N is on the line we have point $(7, 4)$

$\Rightarrow \quad y - 4 = -3(x - 7)$ 1

$y = -3x + 25$ 1

b) The radius of C, r = length NP

P is the intersect of $y = \frac{1}{3}x$ and $y = -3x + 25$

At P : $\frac{1}{3}x = -3x + 25$ (1)

$x = -9x + 75$

$10x = 75 \Rightarrow x = 7.5$

$y = \frac{1}{3}(7.5) = 2.5 \quad \therefore P(7.5, 2.5)$ (1)

Length PN $= \sqrt{(7.5-7)^2 + (4-2.5)^2} = \sqrt{5/2}$ (1)

C : $(x-7)^2 + (y-4)^2 = 5/2$ (1)

c) When $y = \frac{1}{3}x + k$ satisfies the equation for C

$(x-7)^2 + \left(\frac{1}{3}x + k - 4\right)^2 = \frac{5}{2}$

$x^2 - 14x + 49 + \frac{1}{9}x^2 + \frac{k}{3}x - \frac{4}{3}x + \frac{k}{3}x + k^2 - 4k$
$-\frac{4}{3}x - 4k + 16 = \frac{5}{2}$

$\Rightarrow \frac{10}{9}x^2 + \left(\frac{2}{3}k - \frac{50}{3}\right)x + k^2 - 8k + \frac{125}{2} = 0$ (1)

This quadratic must only have one solution, as the tangent only meets the circle once.

$\Rightarrow b^2 - 4ac = 0$

$\Rightarrow \left(\frac{2}{3}k - \frac{50}{3}\right)^2 - 4\left(\frac{10}{9}\right) \times \left(k^2 - 8k + \frac{125}{2}\right) = 0$ (1)

$= \frac{4}{9}k^2 - \frac{200}{9}k + \frac{2500}{9} - \frac{40}{9}k^2 + \frac{320}{9}k - \frac{2500}{9} = 0$

$= 4k^2 - 200k - 40k^2 + 320k = 0$

$\Rightarrow -36k^2 + 120k = 0$

$k = 0$ is the case for line L.

$-36k + 120 = 0$

$k = \frac{120}{36} = \frac{10}{3}$ 1 for the non-zero constant

equation: $y = \frac{1}{3}x + \frac{10}{3}$

3. $$f(x) = 2x^3 + 5x^2 + 2x + 15$$

(a) Use the factor theorem to show that $(x + 3)$ is a factor of $f(x)$.

(2)

(b) Find the constants a, b and c such that

$$f(x) = (x + 3)(ax^2 + bx + c)$$

(2)

(c) Hence show that $f(x) = 0$ has only one real root.

(2)

(d) Write down the real root of the equation $f(x - 5) = 0$

(1)

a) $x + 3 = 0 \quad \therefore x = -3$

Substitute $x = -3$ into $f(x)$

$f(-3) = 2(-3)^3 + 5(-3)^2 + 2(-3) + 15$

$= -54 + 45 - 6 + 15$ (1)

$f(-3) = 0$

$\therefore (x+3)$ is a factor of $f(x)$ since $f(-3) = 0$ (1)

b) $(x+3)(ax^2 + bx + c) \equiv 2x^3 + 5x^2 + 2x + 15$

$x^3 : \quad a = 2$

$x^2 : \quad 3a + b = 5$

$3(2) + b = 5 \quad \therefore b = -1$ (1)

constant : $3c = 15 \quad \therefore c = 5$ (1)

$\therefore f(x) = (x+3)(2x^2 - x + 5)$

c) $f(x) = 0$: $(x+3)(2x^2 - x + 5) = 0$

if $b^2 - 4ac > 0$, 2 real roots
$b^2 - 4ac = 0$, 1 real root
$b^2 - 4ac < 0$, no real root

$x + 3 = 0$

$x = -3$

(only solution)

$b^2 - 4ac = (-1)^2 - 4(2)(5) = -39 < 0$ (1) (1)

$2x^2 - x + 5 = 0$ has no real solutions

d) $f(x) \longrightarrow f(x-5)$ is a translation $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$

$f(x-5) = 0$: $-3 + 5 = 2$

only root from (c)

$\therefore$ $x = 2$ is only real solution to $f(x-5) = 0$ (1)

4. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

Using the substitution $u = \sqrt{x}$ or otherwise, solve

$$6x + 7\sqrt{x} - 20 = 0$$

(4)

$u = \sqrt{x}$ (1)

$x = u^2$

$6x + 7\sqrt{x} - 20 = 0$

Substitute u^2 in place of x :

$6u^2 + 7u - 20 = 0$

$(3u - 4)(2u + 5) = 0$ (1)

$u = \frac{4}{3}$, $u = -\frac{5}{2}$

$\sqrt{x} = \frac{4}{3}$, $\sqrt{x} = -\frac{5}{2}$ (1)

square root of x can not be negative

$x = \frac{16}{9}$ only (1)

5. **In this question you must show detailed reasoning.**

Solutions relying on calculator technology are not acceptable.

The curve C_1 has equation $y = 8 - 10x + 6x^2 - x^3$

The curve C_2 has equation $y = x^2 - 12x + 14$

(a) Verify that when $x = 1$ the curves C_1 and C_2 intersect.

(2)

The curves also intersect when $x = k$.

Given that $k < 0$

(b) use algebra to find the exact value of k.

(5)

a) Substitute $x=1$ into both curve equations:

$y = 8 - 10(1) + 6(1)^2 - (1)^3 = 3$

$y = (1)^2 - 12(1) + 14 = 3$ (1)

C_1 and C_2 meet at $(1,3)$, so they both intercept at $x=1$. (1)

b) when curves intersect:

$8 - 10x + 6x^2 - x^3 = x^2 - 12x + 14$ (1)

$x^3 - 5x^2 - 2x + 6 = 0$

$$\begin{array}{r} x^2 - 4x - 6 \\ x-1 \overline{) x^3 - 5x^2 - 2x + 6} \\ -\ \ x^3 - x^2 \qquad\qquad \\ \hline -4x^2 - 2x \quad \\ -\ \ -4x^2 + 4x \quad \\ \hline -6x + 6 \\ -\ \ -6x + 6 \\ \hline \end{array}$$

from (a), $(x-1)$ is a factor of the cubic. Hence,

$(x-1)(x^2 - 4x - 6) = 0$

(1) (1)

By using quadratic formula:

$(x-1)\boxed{(x^2-4x-6)}=0$ 1

$x = \dfrac{4 \pm \sqrt{(-4)^2 - 4(1)(-6)}}{2(1)}$

$= \dfrac{4 \pm \sqrt{40}}{2}$

$= \dfrac{4}{2} \pm \dfrac{\sqrt{4} \times \sqrt{10}}{2}$

$x = 2 \pm \sqrt{10}$

Since $k < 0$, the only solution is $x = 2 - \sqrt{10}$.

1

6. Given that

$$f(x) = x^2 - 4x + 5 \qquad x \in \mathbb{R}$$

(a) express f(x) in the form $(x + a)^2 + b$ where a and b are integers to be found.

(2)

The curve with equation $y = f(x)$

- meets the y-axis at the point P
- has a minimum turning point at the point Q

(b) Write down

(i) the coordinates of P

(ii) the coordinates of Q

(2)

a) $f(x) = x^2 - 4x + 5$
$= (x-2)^2 - 4 + 5$ ①
$= (x-2)^2 + 1$ ①

b) meets y-axis when $x = 0$
$f(0) = (0-2)^2 + 1 = 4 + 1 = 5$

(i) $P(0, 5)$ ①

minimum turning point:
min $f(x) = (x-2)^2 + 1$
happens when $(x-2)^2 = 0 \Rightarrow x = 2$
$f(2) = 1$

(ii) $Q(2, 1)$ ①

7.

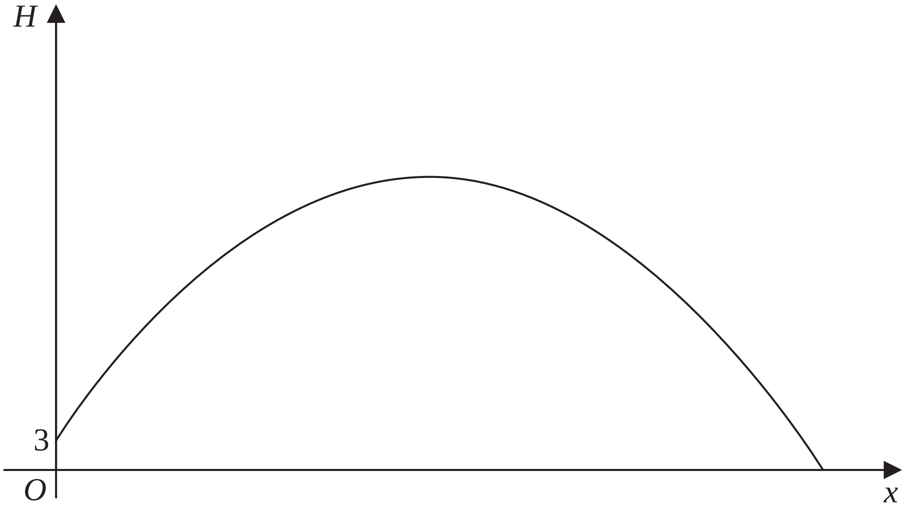

Figure 3

Figure 3 is a graph of the trajectory of a golf ball after the ball has been hit until it first hits the ground.

The vertical height, H metres, of the ball above the ground has been plotted against the horizontal distance travelled, x metres, measured from where the ball was hit.

The ball is modelled as a particle travelling in a vertical plane above horizontal ground.

Given that the ball

- is hit from a point on the top of a platform of vertical height 3 m above the ground
- reaches its maximum vertical height after travelling a horizontal distance of 90 m
- is at a vertical height of 27 m above the ground after travelling a horizontal distance of 120 m

Given also that H is modelled as a **quadratic** function in x

(a) find H in terms of x

(5)

(b) Hence find, according to the model,

(i) the maximum vertical height of the ball above the ground,

(ii) the horizontal distance travelled by the ball, from when it was hit to when it first hits the ground, giving your answer to the nearest metre.

(3)

(c) The possible effects of wind or air resistance are two limitations of the model. Give one other limitation of this model.

(1)

a) info given: 1. contains $(0, 3) \Rightarrow$ H-intercept 3
2. turning point at $x = 90$
3. contains $(120, 27)$

let $H = ax^2 + bx + c$

1. contains $(0, 3)$

$3 = a(0)^2 + b(0) + c$
$c = 3$ ①

2. turning point at $x = 90$

$H = ax^2 + bx + 3$
$\frac{dH}{dx} = 2ax + b$ ①

$2ax + b = 0$ when $x = 90$
$180a + b = 0$ ①
①

3. contains $(120, 27)$

$27 = a(120)^2 + b(120) + 3$
$24 = 14400a + 120b$ ② ①

solve ① and ② simultaneously using calculator:

$a = -\frac{1}{300}$, $b = \frac{3}{5}$

$H = -\frac{1}{300}x^2 + \frac{3}{5}x + 3$ ①

b) (i) maximum vertical height is at $x = 90$

$H = -\frac{1}{300}(90)^2 + \frac{3}{5}(90) + 3$
$= 30\text{m}$ ①

(ii) find roots of H, i.e. when $H = 0$

$-0.03x^2 + 0.6x + 3 = 0$ ①
$x = -4.868...$, $x = 184.868...$

horizontal distance is 185m (nearest metre) ①

c) The ball's dimensions are not considered. ①

8. On a roller coaster ride, passengers travel in carriages around a track.

On the ride, carriages complete multiple circuits of the track such that

- the maximum vertical height of a carriage above the ground is 60 m
- a carriage starts a circuit at a vertical height of 2 m above the ground
- the ground is horizontal

The vertical height, H m, of a carriage above the ground, t seconds after the carriage starts the first circuit, is modelled by the equation

$$H = a - b(t - 20)^2$$

where a and b are positive constants.

(a) Find a complete equation for the model. **(3)**

(b) Use the model to determine the height of the carriage above the ground when $t = 40$ **(1)**

In an alternative model, the vertical height, H m, of a carriage above the ground, t seconds after the carriage starts the first circuit, is given by

$$H = 29\cos(9t + \alpha)° + \beta \qquad 0 \leqslant \alpha < 360°$$

where α and β are constants.

(c) Find a complete equation for the alternative model. **(2)**

Given that the carriage moves continuously for 2 minutes,

(d) give a reason why the alternative model would be more appropriate. **(1)**

a) $H = a - b(t-20)^2$

given:
1. turning point is when $H = 60$
2. when $t = 0$, $H = 2$

turning point has $H = 60$ and $t = 20$ $\therefore$ $60 = a + b(20-20)$
$a = 60$ (1)

sub in $t = 0$, $h = 2$:
(1)
$2 = 60 - b(0-20)^2 \Rightarrow 2 = 60 - 400b$

$400b = 58$

$b = 0.145$

$H = 60 - 0.145(t-20)^2$ (1)

b) sub in $t = 40$

$H = 60 - 0.145(40-20)^2$

$= 2\text{ m}$ (1)

c) $H = 29\cos(9t + \alpha) + \beta$

$\frac{dH}{dt} = -261\sin(9t + \alpha) = 0$ when $t = 20$

$\therefore \sin(180 + \alpha) = 0$

$180 + \alpha = 0 \Rightarrow \alpha = -180°$, out of range

$180 + \alpha = 360 \Rightarrow \alpha = 180°$, in range (1)

$H = 29\cos(9t + 180) + \beta$ $(0 \leq \alpha < 360)$

sub in $t = 0$, $H = 2$

$2 = 29\cos 180 + \beta$

$2 = -29 + \beta$

$\beta = 31$

$H = 29\cos(9t + 180) + 31$ (1)

d) The alternative model allows for more than one circuit (1)